

# SL Paper 1

- a. Show that  $2^n \equiv (-1)^n \pmod{3}$ , where  $n \in \mathbb{N}$ . [3]
- b. Hence show that an integer is divisible by 3 if and only if the difference between the sum of its binary (base 2) digits in even-numbered positions [3] and the sum of its binary digits in odd-numbered positions is divisible by 3.
- c. Express the hexadecimal (base 16) number  $ABBA_{16}$  in binary. [4]

## Markscheme

### a. METHOD 1

$$2^n = (3 - 1)^n \quad \mathbf{M1}$$

$$= 3^n + n3^{n-1}(-1) + \frac{n(n-1)}{2}3^{n-2}(-1)^2 + \dots + (-1)^n \quad \mathbf{A1}$$

since all terms apart from the last one are divisible by 3  $\mathbf{R1}$

$$2^n \equiv (-1)^n \pmod{3} \quad \mathbf{AG}$$

### METHOD 2

attempt to reduce the powers of  $2 \pmod{3}$   $\mathbf{M1}$

$$2^0 = 1 \pmod{3}; 2^1 = -1 \pmod{3}; 2^2 = 1 \pmod{3}; 2^3 = -1 \pmod{3} \dots \quad \mathbf{A1}$$

since  $1 \pmod{3} \times 2 = -1 \pmod{3}$  and  $-1 \pmod{3} \times 2 = 1 \pmod{3}$  the result can be generalized  $\mathbf{R1}$

$$2^n \equiv (-1)^n \pmod{3} \quad \mathbf{AG}$$

[3 marks]

- b. the binary number  $N = (a_n a_{n-1} \dots a_2 a_1 a_0)_2$  has numerical value

$$a_0 \times 1 + a_1 \times 2 + a_2 \times 2^2 + \dots + a_n \times 2^n \quad \mathbf{A1}$$

$$N = (a_0 - a_1 + a_2 - \dots (-1)^n a_n) \pmod{3} \quad \mathbf{M1A1}$$

hence divisibility of  $N$  by 3 coincides with statement of question  $\mathbf{AG}$

[3 marks]

- c.  $ABBA_{16} = 10 \times 16^3 + 11 \times 16^2 + 11 \times 16 + 10 \times 1 \quad (\mathbf{A1})$

$$N = (1010)_2 \times 2^{12} + (1011)_2 \times 2^8 + (1011)_2 \times 2^4 + (1010)_2 \times 2^0 \quad (\mathbf{M1})(\mathbf{A1})$$

**Note:** Award  $\mathbf{M1}$  for expressing A and B in binary.

$$N = (1010101110111010)_2 \quad \mathbf{A1}$$

[4 marks]

## Examiners report

- a. Many candidates were able to make a beginning to this question and attempted a solution to part (a). Some were let down by being unable to fully explain their reasoning.
- b. In part (b) a number of fully correct answers were seen but some candidates appeared to be completely unaware of what constituted a sensible approach.
- c. Again in part (c) a number of correct answers were seen but a significant number also appeared to have little idea on how to start.

Consider the recurrence relation  $H_{n+1} = 2H_n + 1$ ,  $n \in \mathbb{Z}^+$  where  $H_1 = 1$ .

- a. Find  $H_2$ ,  $H_3$  and  $H_4$ . [2]
- b. Conjecture a formula for  $H_n$  in terms of  $n$ , for  $n \in \mathbb{Z}^+$ . [1]
- c. Prove by mathematical induction that your formula is valid for all  $n \in \mathbb{Z}^+$ . [5]

## Markscheme

a.  $H_2 = 2H_1 + 1$  (M1)

$$= 3; H_3 = 7; H_4 = 15 \quad \mathbf{A1}$$

[2 marks]

b.  $H_n = 2^n - 1$  A1

[1 mark]

c. let  $P(n)$  be the proposition that  $H_n = 2^n - 1$  for  $n \in \mathbb{Z}^+$

from (a)  $H_1 = 1 = 2^1 - 1$  A1

so  $P(1)$  is true

assume  $P(k)$  is true for some  $k \Rightarrow H_k = 2^k - 1$  M1

then  $H_{k+1} = 2H_k + 1$  (M1)

$$= 2 \times (2^k - 1) + 1 \quad \mathbf{A1}$$

$$= 2^{k+1} - 1$$

$P(1)$  is true and  $P(k)$  is true  $\Rightarrow P(k+1)$  is true, hence  $P(n)$  is true for all  $n \in \mathbb{Z}^+$  by the principle of mathematical induction R1

**Note:** Only award the R1 if all earlier marks have been awarded.

[5 marks]

## Examiners report

- a. This was a successful question for many candidates with many wholly correct answers seen. The vast majority of candidates were able to answer parts (a) and (b) and most knew how to start part (c). A few candidates were let down by not realising the need for a degree of formality in the presentation of the inductive proof.
- b. This was a successful question for many candidates with many wholly correct answers seen. The vast majority of candidates were able to answer parts (a) and (b) and most knew how to start part (c). A few candidates were let down by not realising the need for a degree of formality in the presentation of the inductive proof.
- c. This was a successful question for many candidates with many wholly correct answers seen. The vast majority of candidates were able to answer parts (a) and (b) and most knew how to start part (c). A few candidates were let down by not realising the need for a degree of formality in the presentation of the inductive proof.
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Consider the Diophantine equation  $7x - 5y = 1$ ,  $x, y \in \mathbb{Z}$ .

- a. Find the general solution to this equation. [3]
- b. Hence find the solution with minimum positive value of  $xy$ . [2]
- c. Find the solution satisfying  $xy = 2014$ . [3]

## Markscheme

- a. one solution is  $x = -2, y = -3$  (or  $(3, 4)$ ) **(A1)**

the general solution is

$$x = -2 + 5N, y = -3 + 7N \text{ (or } x = 3 + 5M, y = 4 + 7M) \quad \mathbf{M1A1}$$

**[3 marks]**

- b. a listing of small values of the product **(M1)**

$$\Rightarrow x = -2, y = -3 \text{ (the least positive value of } xy \text{ is 6)} \quad \mathbf{A1}$$

**[2 marks]**

- c. use of “table” or otherwise to solve

$$35N^2 - 29N + 6 = 2014 \text{ (or } 35M^2 + 41M + 12 = 2014) \quad \mathbf{(M1)}$$

$$\text{obtain } N = 8 \text{ (or } M = 7) \quad \mathbf{(A1)}$$

$$x = 38, y = 53 \quad \mathbf{A1}$$

**[3 marks]**

## Examiners report

- a. This was also a very successful question with many wholly correct answers seen. A small number of candidates made arithmetic errors in the calculations. Some candidates used unnecessarily long and complex methods for parts (b) and (c) which would have potentially left them short of time elsewhere.
- b. This was also a very successful question with many wholly correct answers seen. A small number of candidates made arithmetic errors in the calculations. Some candidates used unnecessarily long and complex methods for parts (b) and (c) which would have potentially left them short of time elsewhere.
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- a. A number written in base 5 is 4303. Find this as a number written in base 10. [2]
- b. 1000 is a number written in base 10. Find this as a number written in base 7. [5]

## Markscheme

a.  $4303_5 = 4 \times 5^3 + 3 \times 5^2 + 0 \times 5^1 + 3 \times 5^0$  (M1)

$$= 500 + 75 + 3$$

$$= 578 \quad \mathbf{A1}$$

[2 marks]

b. **METHOD 1**

$$1000 = a_0 + 7a_1 + 49a_2 + 343a_3 \quad (\mathbf{M1})$$

(since  $343a_3 < 1000 \Rightarrow a_3 = 2$  (A1)

$$1000 = a_0 + 7a_1 + 49a_2 + 686$$

$$a_0 + 7a_1 + 49a_2 = 314 \Rightarrow a_2 = 6 \quad (\mathbf{A1})$$

$$a_0 + 7a_1 = 20 \Rightarrow a_1 = 2, a_0 = 6 \quad (\mathbf{A1})$$

$$\Rightarrow 1000_{10} = 2626_7 \quad \mathbf{A1}$$

**METHOD 2**

$$1000 = 7 \times 142 + 6 \quad (\mathbf{M1})$$

$$142 = 7 \times 20 + 2 \quad (\mathbf{A1})$$

$$20 = 7 \times 2 + 6 \quad (\mathbf{A1})$$

$$2 = 7 \times 0 + 2 \quad (\mathbf{A1})$$

$$\Rightarrow (1000)_{10} = 2626_7 \quad \mathbf{A1}$$

[5 marks]

# Examiners report

- a. [N/A]  
b. [N/A]

The simple connected planar graph  $J$  has the following adjacency table.

	A	B	C	D	E	F	G	H
A	–	0	1	1	1	0	0	0
B	0	–	1	1	1	1	1	1
C	1	1	–	1	1	0	1	1
D	1	1	1	–	1	0	0	0
E	1	1	1	1	–	1	1	0
F	0	1	0	0	1	–	1	0
G	0	1	1	0	1	1	–	1
H	0	1	1	0	0	0	1	–

- a.i. Without attempting to draw  $J$ , verify that  $J$  satisfies the handshaking lemma; [1]
- a.ii. Without attempting to draw  $J$ , determine the number of faces in  $J$ . [3]
- b. The vertices D and G are joined by a single edge to form the graph  $K$ . Show that  $K$  is not planar. [3]
- c.i. Explain why a graph containing a cycle of length three cannot be bipartite. [3]
- c.ii. Hence by finding a cycle of length three, show that the complement of  $K$  is not bipartite. [2]

## Markscheme

- a.i. the sum of degrees of the vertices is even (36) or the sum of degrees of the vertices is twice the number of edges **A1**

[2 marks]

- a.ii. the number of edges ( $e$ ) is 18 **A1**

using Euler's relation  $v - e + f = 2$  **M1**

$$f = 2 + 18 - 8 = 12 \quad \mathbf{A1}$$

[2 marks]

- b. if  $K$  is planar then  $e \leq 3v - 6$  **M1**

$$v = 8 \text{ and } e = 19 \quad \mathbf{A1}$$

the inequality is not satisfied so  $K$  is not planar **A1AG**

[3 marks]

- c.i. let PQRP be a cycle of length 3 in a graph **M1**

**Note:** Accept a diagram instead of this statement.

suppose the graph is bipartite

then P must belong to one of the two disjoint sets of vertices and Q, R must belong to the other disjoint set **R1**

but Q, R cannot belong to the same set because they are linked **R1**

therefore the graph cannot be bipartite **AG**

**[???** marks]

c.ii. for example, a suitable cycle of order 3 is AFHA **(M1)A1**

**Note:** Award **M1** for a valid attempt at drawing the complement or constructing its adjacency table.

**[???** marks]

## Examiners report

a.i. [N/A]

a.ii. [N/A]

b. [N/A]

c.i. [N/A]

c.ii. [N/A]

a. Use the Euclidean algorithm to find the greatest common divisor of 74 and 383. [4]

b. Hence find integers  $s$  and  $t$  such that  $74s + 383t = 1$ . [5]

## Markscheme

a.  $383 = 5 \times 74 + 13$  **M1**

$$74 = 5 \times 13 + 9 \quad \mathbf{A1}$$

$$13 = 1 \times 9 + 4 \quad \mathbf{(A1)}$$

$$9 = 2 \times 4 + 1$$

$$4 = 4 \times 1 + 0$$

$$\Rightarrow \gcd(74, 383) = 1 \quad \mathbf{A1}$$

**[4 marks]**

b. **EITHER**

$$1 = 9 - 2 \times 4 \quad \mathbf{(M1)}$$

$$= 9 - 2(13 - 1 \times 9) = 3 \times 9 - 2 \times 13 \quad \mathbf{(A1)}$$

$$= 3(74 - 5 \times 13) - 2 \times 13 = 3 \times 74 - 17 \times 13 \quad \mathbf{(A1)}$$

$$= 3 \times 74 - 17(383 - 5 \times 74) = 88 \times 74 - 17 \times 383$$

**OR**

$$13 = 383 - 5 \times 74 \quad (M1)$$

$$9 = 74 - 5 \times 13$$

$$= 74 - 5(383 - 5 \times 74)$$

$$= 26 \times 74 - 5 \times 383 \quad (A1)$$

$$4 = 13 - 9$$

$$= (383 - 5 \times 74) - (26 \times 74 - 5 \times 383)$$

$$= 6 \times 383 - 31 \times 74 \quad (A1)$$

$$1 = 9 - 2 \times 4$$

$$= (26 \times 74 - 5 \times 383) - 2(6 \times 383 - 31 \times 74)$$

$$= 88 \times 74 - 17 \times 383$$

**THEN**

$$\Rightarrow s = 88 \text{ and } t = -17 \quad \mathbf{A1A1}$$

**[5 marks]**

## Examiners report

a. [N/A]

b. [N/A]

a. Given that  $a \equiv b \pmod{p}$ , show that  $a^n \equiv b^n \pmod{p}$  for all  $n \in \mathbb{Z}^+$ . [4]

b. Show that  $29^{13} + 13^{29}$  is exactly divisible by 7. [5]

## Markscheme

a.  $a \equiv b \pmod{p} \Rightarrow a = b + pN, N \in \mathbb{Z} \quad \mathbf{M1}$

$$a^n = (b + pN)^n = b^n + nb^{n-1}pN \dots \quad \mathbf{M1A1}$$

$$= b^n + pM \text{ where } M \in \mathbb{Z} \quad \mathbf{A1}$$

this shows that  $a^n \equiv b^n \pmod{p} \quad \mathbf{AG}$

**[4 marks]**

b.  $29 \equiv 1 \pmod{7} \Rightarrow 29^{13} \equiv 1^{13} \equiv 1 \pmod{7} \quad \mathbf{M1A1}$

$$13 \equiv -1 \pmod{7} \Rightarrow 13^{29} \equiv (-1)^{29} \equiv -1 \pmod{7} \quad \mathbf{A1}$$

$$\text{therefore } 29^{13} + 13^{29} \equiv 1 + (-1) \equiv 0 \pmod{7} \quad \mathbf{M1A1}$$

this shows that  $29^{13} + 13^{29}$  is exactly divisible by 7  $\mathbf{AG}$

**[5 marks]**

## Examiners report

a. [N/A]

b. [N/A]

a. Find the general solution to the Diophantine equation  $3x + 5y = 7$ . [5]

- b. Find the values of  $x$  and  $y$  satisfying the equation for which  $x$  has the smallest positive integer value greater than 50.

[2]

## Markscheme

- a. by any method including trial and error or the Euclidean algorithm, a specific solution is, for example,  $x = 4$ ,  $y = -1$  **(A1)(A1)**

$$3(4) + 5(-1) = 7 \quad (\text{equation i})$$

$$3x + 5y = 7 \quad (\text{equation ii})$$

$$\text{equation ii} - \text{equation i: } 3(x - 4) + 5(y + 1) = 0$$

$$\frac{4-x}{5} = \frac{y+1}{3} = N \quad (\mathbf{M1})$$

$$x = 4 - 5N \quad \mathbf{A1}$$

$$y = 3N - 1 \quad \mathbf{A1}$$

- b. smallest positive integer  $> 50$  occurs when  $N = -10$  **(M1)**

$$x = 54, y = -31 \quad \mathbf{A1}$$

## Examiners report

- a. This question was well answered in general although a minority appeared not to realise that Diophantine means a solution in integers. It was expected that candidates would find a particular solution by inspection but some took a little longer by going via the Euclidean Algorithm.
- b. This question was well answered in general although a minority appeared not to realise that Diophantine means a solution in integers. It was expected that candidates would find a particular solution by inspection but some took a little longer by going via the Euclidean Algorithm.

- a. Consider the linear congruence  $ax \equiv b \pmod{p}$  where  $a, b \in \mathbb{Z}^+$ ,  $p$  is a prime and  $\gcd(a, p) = 1$ . Using Fermat's little theorem, show that  $x \equiv a^{p-2}b \pmod{p}$ . [3]

- b. **Hence** find the smallest value of  $x$  greater than 100 satisfying the linear congruence  $3x \equiv 13 \pmod{19}$ . [4]

## Markscheme

- a. multiplying both sides by  $a^{p-2}$ , **M1**

$$a^{p-1}x \equiv a^{p-2}b \pmod{p} \quad \mathbf{A1}$$

$$\text{using } a^{p-1} \equiv 1 \pmod{p} \quad \mathbf{R1}$$

$$\text{therefore, } x \equiv a^{p-2}b \pmod{p} \quad \mathbf{AG}$$

**[3 marks]**

- b. using the above result,

$$x \equiv 3^{17} \times 13 \pmod{19} \pmod{19} \pmod{19} \equiv 16\,7882\,2119 \pmod{19} \quad \mathbf{A1}$$

$$\equiv 17 \pmod{19} \quad (\mathbf{M1})\mathbf{A1}$$



$$x = 112 \quad \mathbf{A1}$$

[4 marks]

## Examiners report

- a. [N/A]  
b. [N/A]
- 

- a. Use the Euclidean algorithm to find  $\gcd(162, 5982)$ . [4]
- b. The relation  $R$  is defined on  $\mathbb{Z}^+$  by  $nRm$  if and only if  $\gcd(n, m) = 2$ . [7]
- (i) By finding counterexamples show that  $R$  is neither reflexive nor transitive.
- (ii) Write down the set of solutions of  $nR6$ .

## Markscheme

a.  $5982 = 162 \times 36 + 150 \quad \mathbf{M1A1}$

$$162 = 150 \times 1 + 12 \quad \mathbf{A1}$$

$$150 = 12 \times 12 + 6$$

$$12 = 6 \times 2 + 0 \Rightarrow \gcd \text{ is } 6 \quad \mathbf{A1}$$

[4 marks]

b. (i) for example,  $\gcd(4, 4) = 4 \quad \mathbf{A1}$

$$4 \neq 2 \quad \mathbf{R1}$$

so  $R$  is not reflexive  $\mathbf{AG}$

for example

$$\gcd(4, 2) = 2 \text{ and } \gcd(2, 8) = 2 \quad \mathbf{M1A1}$$

$$\text{but } \gcd(4, 8) = 4 (\neq 2) \quad \mathbf{R1}$$

so  $R$  is not transitive  $\mathbf{AG}$

(ii) **EITHER**

even numbers  $\mathbf{A1}$

not divisible by 6  $\mathbf{A1}$

**OR**

$$\{2 + 6n : n \in \mathbb{N}\} \cup \{4 + 6n : n \in \mathbb{N}\} \quad \mathbf{A1A1}$$

**OR**

$$2, 4, 8, 10, \dots \quad \mathbf{A2}$$

[7 marks]

## Examiners report

- a. This was a successful question for many students with many wholly correct answers seen. Part (a) was successfully answered by most candidates and those candidates usually had a reasonable understanding of how to complete part (b). A number were not fully successful in knowing how to explain their results.
- b. This was a successful question for many students with many wholly correct answers seen. Part (a) was successfully answered by most candidates and those candidates usually had a reasonable understanding of how to complete part (b). A number were not fully successful in knowing how to explain their results.

- a. Express the number 47502 as a product of its prime factors. [2]
- b. The positive integers  $M$ ,  $N$  are such that  $\gcd(M, N) = 63$  and  $\text{lcm}(M, N) = 47502$ . Given that  $M$  is even and  $M < N$ , find the two possible pairs of values for  $M$ ,  $N$ . [5]

## Markscheme

a.

$$\begin{array}{r}
 2) \overline{47502} \\
 3) \overline{23751} \\
 3) \overline{7917} \\
 7) \overline{2639} \\
 13) \overline{377} \\
 \quad \underline{29}
 \end{array}
 \quad (M1)$$

therefore  $47502 = 2 \times 3^2 \times 7 \times 13 \times 29$  *A1*

[2 marks]

- b. noting that  $MN = \gcd \times \text{lcm} = 2 \times 3^4 \times 7^2 \times 13 \times 29$  *(M1)*

the possibilities are

$$(M, N) = (126, 23751) \quad \textit{A1A1}$$

$$(M, N) = (1638, 1827) \quad \textit{A1A1}$$

[5 marks]

## Examiners report

- a. As expected, the factorization in (a) was successfully completed by most candidates.
- b. Part (b) caused problems for some candidates. The most common mistake was that only one pair of values for  $M$ ,  $N$  was given.

- a. Solve the recurrence relation  $u_n = 4u_{n-1} - 4u_{n-2}$  given that  $u_0 = u_1 = 1$ .

[6]

b. Consider  $v_n$  which satisfies the recurrence relation  $2v_n = 7v_{n-1} - 3v_{n-2}$  subject to the initial conditions  $v_0 = v_1 = 1$ .

Prove by using strong induction that  $v_n = \frac{4}{5}\left(\frac{1}{2}\right)^n + \frac{1}{5}(3)^n$  for  $n \in \mathbb{N}$ .

## Markscheme

a. auxiliary equation is  $m^2 - 4m + 4 = 0$  **M1A1**

hence  $m$  has a repeated root of 2 **(A1)**

solution is of the form  $u_n = a(2)^n + bn(2)^n$  **M1**

using the initial conditions **M1**

$$\Rightarrow a = 1 \text{ and } b = -\frac{1}{2}$$

$$\Rightarrow u_n = 2^n - \frac{n}{2}(2)^n \quad \mathbf{A1}$$

**[6 marks]**

b.  $v_n = \frac{4}{5}\left(\frac{1}{2}\right)^n + \frac{1}{5}(3)^n$

$$\text{let } n = 0 \quad v_0 = \frac{4}{5}\left(\frac{1}{2}\right)^0 + \frac{1}{5}(3)^0 = \frac{4}{5} + \frac{1}{5} = 1$$

$$\text{let } n = 1 \quad v_1 = \frac{4}{5}\left(\frac{1}{2}\right)^1 + \frac{1}{5}(3)^1 = \frac{2}{5} + \frac{3}{5} = 1$$

hence true for  $n = 0$  and  $n = 1$  **M1A1**

assume that  $v_j = \frac{4}{5}\left(\frac{1}{2}\right)^j + \frac{1}{5}(3)^j$  is true for all  $j > k + 1$  **M1**

hence  $v_k = \frac{4}{5}\left(\frac{1}{2}\right)^k + \frac{1}{5}(3)^k$  and  $v_{k-1} = \frac{4}{5}\left(\frac{1}{2}\right)^{k-1} + \frac{1}{5}(3)^{k-1}$

$$v_{k+1} = \frac{7v_k - 3v_{k-1}}{2}$$

$$v_{k+1} = \frac{7\left[\frac{4}{5}\left(\frac{1}{2}\right)^k + \frac{1}{5}(3)^k\right] - 3\left[\frac{4}{5}\left(\frac{1}{2}\right)^{k-1} + \frac{1}{5}(3)^{k-1}\right]}{2} \quad \mathbf{M1A1}$$

$$v_{k+1} = \frac{7\left[\frac{8}{5}\left(\frac{1}{2}\right)^{k+1} + \frac{1}{15}(3)^{k+1}\right] - 3\left[\frac{16}{5}\left(\frac{1}{2}\right)^{k+1} + \frac{1}{45}(3)^{k+1}\right]}{2} \quad \mathbf{(A1)}$$

$$v_{k+1} = \frac{\frac{56}{5}\left(\frac{1}{2}\right)^{k+1} + \frac{7}{15}(3)^{k+1} - \frac{48}{5}\left(\frac{1}{2}\right)^{k+1} - \frac{1}{15}(3)^{k+1}}{2} \quad \mathbf{(A1)}$$

$$v_{k+1} = \frac{\frac{8}{5}\left(\frac{1}{2}\right)^{k+1} + \frac{6}{15}(3)^{k+1}}{2} \quad \mathbf{(A1)}$$

**Note:** Only one of the above **(A1)** can be implied.

$$v_{k+1} = \frac{4}{5}\left(\frac{1}{2}\right)^{k+1} + \frac{1}{15}(3)^{k+1}$$

since the basis step and the inductive step have been verified, the Principle of Mathematical Induction tells us that  $v_n = \frac{4}{5}\left(\frac{1}{2}\right)^n + \frac{1}{5}(3)^n$  is the general solution **R1**

**[9 marks]**

**Note:** Only award final **R1** if at least 5 previous marks have been awarded.

## Examiners report

a. [N/A]

b. [N/A]

Prove that  $3k + 2$  and  $5k + 3$ ,  $k \in \mathbb{Z}$  are relatively prime.

## Markscheme

### EITHER

$3k + 2$  and  $5k + 3$ ,  $k \in \mathbb{Z}$  are relatively prime

if, for all  $k$ , there exist  $m, n \in \mathbb{Z}$  such that

$$m(3k + 2) + n(5k + 3) = 1 \quad \text{RIMIAI}$$

$$\Rightarrow 3m + 5n = 0 \quad \text{AI}$$

$$2m + 3n = 1 \quad \text{AI}$$

$$m = 5, n = -3 \quad \text{AI}$$

hence they are relatively prime  $\quad \text{AG}$

### OR

$$5k + 3 = 1 \times (3k + 2) + 2k + 1 \quad \text{MIAI}$$

$$3k + 2 = 1 \times (2k + 1) + k + 1 \quad \text{AI}$$

$$2k + 1 = 1 \times (k + 1) + k \quad \text{AI}$$

$$k + 1 = 1 \times (k) + 1 \quad \text{AI}$$

$$\text{GDC} = 1 \quad \text{AI}$$

so  $5k + 3$  and  $3k + 2$  are relatively prime  $\quad \text{AG}$

[6 marks]

## Examiners report

This was often done using Euclid's algorithm rather than writing  $m(3k + 2) + n(5k + 3) = 1$  for the relatively prime numbers.

a. Prove that the number 14641 is the fourth power of an integer in any base greater than 6. [3]

b. For  $a, b \in \mathbb{Z}$  the relation  $aRb$  is defined if and only if  $\frac{a}{b} = 2^k$ ,  $k \in \mathbb{Z}$ . [8]

(i) Prove that  $R$  is an equivalence relation.

(ii) List the equivalence classes of  $R$  on the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ .

## Markscheme

a.  $14641$  (base  $a > 6$ ) =  $a^4 + 4a^3 + 6a^2 + 4a + 1$ ,  $\quad \text{MIAI}$

$$= (a + 1)^4 \quad \text{AI}$$

this is the fourth power of an integer  $\quad \text{AG}$

[3 marks]

b. (i)  $aRa$  since  $\frac{a}{a} = 1 = 2^0$ , hence  $R$  is reflexive **AI**

$$aRb \Rightarrow \frac{a}{b} = 2^k \Rightarrow \frac{b}{a} = 2^{-k} \Rightarrow bRa$$

so  $R$  is symmetric **AI**

$$aRb \text{ and } bRc \Rightarrow \frac{a}{b} = 2^m, m \in \mathbb{Z} \text{ and } bRc \Rightarrow \frac{b}{c} = 2^n, n \in \mathbb{Z} \quad \mathbf{MI}$$

$$\Rightarrow \frac{a}{b} \times \frac{b}{c} = \frac{a}{c} = 2^{m+n}, m+n \in \mathbb{Z} \quad \mathbf{AI}$$

$\Rightarrow aRc$  so transitive **RI**

hence  $R$  is an equivalence relation **AG**

(ii) equivalence classes are  $\{1, 2, 4, 8\}$ ,  $\{3, 6\}$ ,  $\{5, 10\}$ ,  $\{7\}$ ,  $\{9\}$  **A3**

**Note:** Award **A2** if one class missing, **A1** if two classes missing, **A0** if three or more classes missing.

[8 marks]

## Examiners report

a. This was not difficult but a surprising number of candidates were unable to do it. Care with notation and logic were lacking.

b. The question was at first straightforward but some candidates mixed up the properties of an equivalence relation with those of a group. The idea of an equivalence class is still not clearly understood by many candidates so that some were missing.

---

a. Using mathematical induction or otherwise, prove that the number  $(1020)_n$ , that is the number 1020 written with base  $n$ , is divisible by 3 [8]  
for all values of  $n$  greater than 2.

b. Explain briefly why the case  $n = 2$  has to be excluded. [1]

## Markscheme

a.  $(1020)_n = n^3 + 2n$  **(RI)**

so we are required to prove that  $n^3 + 2n$  is divisible by 3 for  $n \geq 3$  **(RI)**

**EITHER**

when  $n = 3$ ,  $n^3 + 2n = 33$  which is divisible by 3 so the result is true for  $n = 3$  **AI**

assume the result is true for  $n = k$ , i.e.  $k^3 + 2k$  is divisible by 3 **MI**

for  $n = k + 1$ ,

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 \quad \mathbf{MI}$$

$$= (k^3 + 2k) + 3(k^2 + k + 1) \quad \mathbf{AI}$$

the second term is clearly divisible by 3 and the first term is divisible by 3 by hypothesis **AI**

therefore true for  $n = k \Rightarrow$  true for  $n = k + 1$  and since shown true for  $n = 3$ ,

the result is proved by induction **RI**

**Note:** Award the final **RI** only if the two **MI** marks have been awarded.

**OR**

there are three cases to consider, let  $N$  be a positive integer

case 1:  $n = 3N$ , in this case

$$n(n^2 + 2) = 3N(9N^2 + 2) \text{ which is divisible by } 3 \quad \mathbf{MIAI}$$

case 2:  $n = 3N + 1$ , in this case,

$$n(n^2 + 2) = (3N + 1)(9N^2 + 6N + 3) \text{ which is divisible by } 3 \quad \mathbf{MIAI}$$

case 3:  $n = 3N + 2$ , in this case,

$$n(n^2 + 2) = (3N + 2)(9N^2 + 12N + 6) \text{ which is divisible by } 3 \quad \mathbf{MIAI}$$

this proves the required result for all  $n > 2$

**[8 marks]**

b. numbers to base 2 do not use the digit 2 or equivalent **RI**

**[1 mark]**

## Examiners report

- a. A significant number of candidates struggled with this question, but a number of wholly correct answers were seen. The majority of candidates tried to use a method of proof by induction although a number got lost in trying to establish the result for  $k + 1$ . Again the presentation and explanation of some of the solutions was poor. Some excellent solutions were seen using other methods including one which noted simply that all positive integers  $n$  are either 0 or  $\pm 1$  modulo 3 so that  $n(n^2 + 2) \equiv 0$  modulo 3 for all  $n$ .
- b. A significant number of candidates struggled with this question, but a number of wholly correct answers were seen. The majority of candidates tried to use a method of proof by induction although a number got lost in trying to establish the result for  $k + 1$ . Again the presentation and explanation of some of the solutions was poor. Some excellent solutions were seen using other methods including one which noted simply that all positive integers  $n$  are either 0 or  $\pm 1$  modulo 3 so that  $n(n^2 + 2) \equiv 0$  modulo 3 for all  $n$ .

---

a. Prove that if  $\gcd(a, b) = 1$  and  $\gcd(a, c) = 1$ , then  $\gcd(a, bc) = 1$ . [5]

b. (i) A simple graph has  $e$  edges and  $v$  vertices, where  $v > 2$ . Prove that if all the vertices have degree at least  $k$ , then  $2e \geq kv$ . [6]

(ii) **Hence** prove that every planar graph has at least one vertex of degree less than 6.

## Markscheme

a. **EITHER**

since  $\gcd(a, b) = 1$  and  $\gcd(a, c) = 1$  then

$$ax + by = 1 \text{ and } ap + cq = 1 \text{ for } x, y, p, q \in \mathbb{Z} \quad \mathbf{MIAI}$$

hence

$$(ax + by)(ap + cq) = 1 \quad \mathbf{AI}$$

$$a(xap + xcq + byp) + bc(yq) = 1 \quad \mathbf{MI}$$

since  $(xap + xcq + byp)$  and  $(yq)$  are integers  $\mathbf{RI}$

$$\text{then } \gcd(a, bc) = 1 \quad \mathbf{AG}$$

**OR**

if  $\gcd(a, bc) \neq 1$ , some prime  $p$  divides  $a$  and  $bc$   $\mathbf{MIAI}$

$p$  divides  $b$  or  $c$   $\mathbf{MI}$

either  $\gcd(a, b) \neq 1$  or  $\gcd(a, c) \neq 1$   $\mathbf{AI}$

contradiction  $\Rightarrow \gcd(a, bc) = 1$   $\mathbf{RI}$

**[5 marks]**

b. (i) each edge contributes 2 to the degree sum  $\mathbf{RI}$

$$\text{so } 2e = \text{degree sum} \quad \mathbf{RI}$$

$$\text{so } 2e \geq kv \quad \mathbf{AG}$$

(ii)  $k = 6$  so  $2e \geq 6v$   $\mathbf{MI}$

for a planar graph with  $v$  vertices and  $e$  edges,  $e \leq 3v - 6$   $\mathbf{MI}$

$$\text{so } 2e \leq 6v - 12 \quad \mathbf{AI}$$

this is a contradiction so at least one vertex must have degree  $< 6$   $\mathbf{RI}$

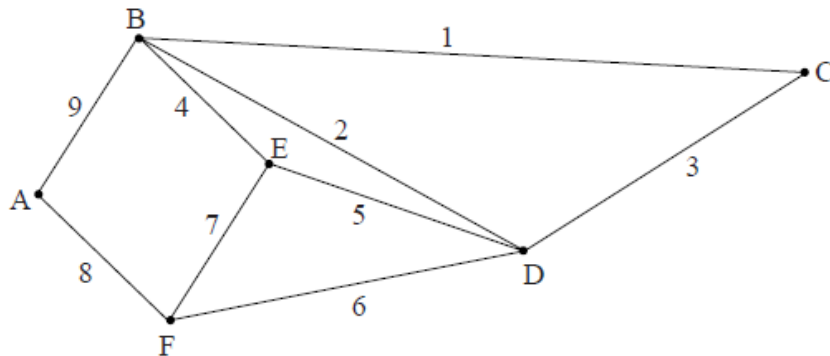
**Note:** Alternative proofs are possible.

**[6 marks]**

## Examiners report

a. A "verbal" argument rather than a symbolic approach was often taken, leading to some doubt about the validity of the "proofs". This discursive approach in trying to prove propositions often leads down blind alleys to confusion.

b. This was reasonably well done although clear proofs were not abundant and the same comments mentioned in (a) apply. In (ii) the word "Hence" was sometimes ignored. Candidates should realize that such "signpost" words are there to help them.



The above diagram shows the weighted graph  $G$ .

a. Determine whether or not  $G$  is bipartite. [2]

b. (i) Write down the adjacency matrix for  $G$ . [4]

(ii) Find the number of distinct walks of length 4 beginning and ending at A.

## Markscheme

a.  $G$  is bipartite *AI*

because it contains a triangle. *RI*

**Note:** Award *RI* for a valid attempt at showing that the vertices cannot be divided into two disjoint sets.

*[2 marks]*

b. (i)

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{AI}$$

(ii) We require the (A, A) element of  $M^4$  which is 13. *MIA2*

*[4 marks]*

## Examiners report

a. [N/A]

b. [N/A]

a. Use the Euclidean Algorithm to show that 275 and 378 are relatively prime. [5]

b. Find the general solution to the diophantine equation  $275x + 378y = 1$ . [7]

## Markscheme

a.  $378 = 1 \times 275 + 103$  *AI*

$$275 = 2 \times 103 + 69 \quad \text{AI}$$

$$103 = 1 \times 69 + 34 \quad \text{AI}$$

$$69 = 2 \times 34 + 1 \quad \text{AI}$$

showing that  $\text{gcd} = 1$ , *i.e.* relatively prime. *RI*

*[5 marks]*

b. Working backwards,

$$1 = 69 - 2 \times (103 - 69) \quad \text{(M1)}$$

$$= 3 \times 69 - 2 \times 103 \quad \text{(A1)}$$



$$= 3 \times (275 - 2 \times 103) - 2 \times 103$$

$$= 3 \times 275 - 8 \times 103 \quad (A1)$$

$$= 3 \times 275 - 8 \times (378 - 275)$$

$$= 11 \times 275 - 8 \times 378 \quad (A1)$$

A solution is  $x = 11, y = -8 \quad (A1)$

The general solution is  $x = 11 + 378n, y = -8 - 275n$  where  $n \in \mathbb{Z} \quad M1A1 \quad N6$

[7 marks]

## Examiners report

a. [N/A]

b. [N/A]

An integer  $N$  given in base  $r$ , can be expressed in base  $s$  in the form

$$N = a_0 + a_1s + a_2s^2 + a_3s^3 + \dots \text{ where } a_0, a_1, a_2, \dots \in \{0, 1, 2, \dots, s-1\}.$$

a. In base 5 an integer is written 1031. Express this integer in base 10. [2]

b. Given that  $N = 365, r = 10$  and  $s = 7$  find the values of  $a_0, a_1, a_2, \dots$  [2]

c. (i) Given that  $N = 899, r = 10$  and  $s = 12$  find the values of  $a_0, a_1, a_2, \dots$  [3]

(ii) Hence write down the integer in base 12, which is equivalent to 899 in base 10.

d. Show that 121 is always a square number in any base greater than 2. [3]

## Markscheme

a.  $1031 = 1 \times 5^3 + 0 \times 5^2 + 3 \times 5 + 1 \quad M1$

$$= 125 + 0 + 15 + 1$$

$$= 141 \quad A1$$

b.  $365 = 1 \times 7^3 + 0 \times 7^2 + 3 \times 7 + 1 \quad M1$

$$\Rightarrow a_0 = 1, a_1 = 3, a_2 = 0, a_3 = 1 \quad A1$$

c. (i)  $899 = 6 \times 12^2 + 2 \times 12 + 11 \quad M1$

$$\Rightarrow a_0 = 11, a_1 = 2, a_2 = 6 \quad A1$$

(ii)  $(899)_{10} = (62B)_{12}$

(where  $B$  represents the digit in base 12 given by  $a_0 = 11$ )  $A1$

**Note:** Accept any letter in place of  $B$  provided it is defined

d. 121 in base  $r$  is  $1 + 2r + r^2 \quad M1A1$

$$= (r + 1)^2 \quad A1$$

which is a square for all  $r$  **AG**

## Examiners report

- a. This question was very well answered in general, although some candidates failed to see that  $121 = 11^2$  in all number bases greater than 2.
- b. This question was very well answered in general, although some candidates failed to see that  $121 = 11^2$  in all number bases greater than 2.
- c. This question was very well answered in general, although some candidates failed to see that  $121 = 11^2$  in all number bases greater than 2.
- d. This question was very well answered in general, although some candidates failed to see that  $121 = 11^2$  in all number bases greater than 2.

- a. (i) Use the Euclidean algorithm to find  $\gcd(6750, 144)$ . [6]
- (ii) Express your answer in the form  $6750r + 144s$  where  $r, s \in \mathbb{Z}$ .
- b. Consider the base 15 number CBA, where A, B, C represent respectively the digits ten, eleven, twelve. [6]
- (i) Write this number in base 10.
- (ii) Hence express this number as a product of prime factors, writing the factors in base 4.

## Markscheme

a. (i)  $6750 = 46 \times 144 + 126$  **MI AI**

$$144 = 1 \times 126 + 18 \quad \mathbf{AI}$$

$$126 = 7 \times 18$$

$$\gcd(6750, 144) = 18 \quad \mathbf{AI N0}$$

(ii)  $18 = 144 - 1 \times 126$  **(MI)**

$$= 144 - (6750 - 46 \times 144)$$

$$= 47 \times 144 + (-1) \times 6750 \quad \mathbf{AI}$$

**[6 marks]**

b. (i)  $n = 10 + 11 \times 15 + 12 \times 15^2$  **(MI)(AI)**

$$= 2875 \quad \mathbf{AI}$$

(ii)  $2875 = 5^3 \times 23$  **AI**

$$= 11 \times 11 \times 11 \times 113 \text{ in base 4} \quad \mathbf{AI AI}$$

**Note:** **AI** for  $11 \times 11 \times 11$ , **AI** for 113.

**[6 marks]**

# Examiners report

a. Most candidates were able to use the Euclidean algorithm to find a gcd and to express the answer in a different form. All but the weakest candidates were able to make a meaningful start to this question.

b. In part (b) many correct answers were seen and a majority of students were able to use the work they had studied on number bases correctly. All but the weakest candidates were able to make a meaningful start to this question.

b. (i) Sum the series  $\sum_{r=0}^{\infty} x^r$ . [11]

(ii) Hence, using sigma notation, deduce a series for

(a)  $\frac{1}{1+x^2}$  ;

(b)  $\arctan x$  ;

(c)  $\frac{\pi}{6}$  .

c. Show that  $\sum_{n=1}^{100} n! \equiv 3 \pmod{15}$ . [4]

## Markscheme

b. (i)  $\sum_{r=0}^{\infty} x^r = 1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x}$  **AI**

(ii) (a) replacing  $x$  by  $-x^2$  gives **(M1)**

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \dots$$
 **AI**

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$
 **(A1)**

$$= \sum_{r=0}^{\infty} (-1)^r x^{2r}$$
 **AI N2**

(b)  $\arctan x = \int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + c$  **M1A1**

$x = 0 \Rightarrow c = 0$  **AI**

$$\arctan x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{2r+1}$$
 **AI**

(c) by taking  $x = \frac{1}{\sqrt{3}}$  **M1**

$$\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6} = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{\sqrt{3}}\right)^{2r+1}}{2r+1}$$
 **AI**

**[11 marks]**

c.  $\sum_{n=1}^{100} n! = 1! + 2! + 3! + 4! + 5! + \dots$  **M1**

$$= 1 + 2 + 6 + 24 + 120 + \dots$$

$$\equiv 1 + 2 + 6 + 24 + 0 + 0 + 0 + \dots \pmod{15} \quad \mathbf{MIAI} \square\square$$

$$\equiv 33 \pmod{15} \quad \mathbf{AI}$$

$$\equiv 3 \pmod{15} \quad \mathbf{AG}$$

**[4 marks]**

## Examiners report

- b. In part (b) many did not recognize the sum of a simple geometric series to infinity and got involved in some heavy Maclaurin work thus wasting time. The clear instruction "Hence" was ignored by many candidates so that the question became more difficult and time consuming than it should have been.
- c. Part (c) proved not to be as difficult as expected.

The positive integer  $N$  is represented by 4064 in base  $b$  and 2612 in base  $b + 1$ .

- a. Determine the value of  $b$ . [4]
- b. Find the representation of  $N$  [6]
- (i) in base 10;
- (ii) in base 12.

## Markscheme

- a. the equation satisfied by  $b$  is

$$4b^3 + 6b + 4 = 2(b + 1)^3 + 6(b + 1)^2 + (b + 1) + 2 \quad \mathbf{MIAI}$$

$$2b^3 - 12b^2 - 13b - 7 = 0 \quad \mathbf{(AI)}$$

$$b = 7 \quad \mathbf{AI}$$

**[4 marks]**

- b. (i)  $N = 4 \times 7^3 + 6 \times 7 + 4 = 1418$

$$\text{or } N = 2 \times 8^3 + 6 \times 8^2 + 1 \times 8 + 2 = 1418 \quad \mathbf{(MI)AI}$$

(ii) 12 1418  $\mathbf{(MI)}$

$$12 \text{ 118 remainder 2} \quad \mathbf{(AI)}$$

$$12 \text{ 9 remainder A (where A = 10)} \quad \mathbf{(AI)}$$

$$1418 = (9A2)_{12} \quad \mathbf{AI}$$

**Note:** Accept alternative correct methods.

[6 marks]

## Examiners report

- a. [N/A]  
b. [N/A]

---

Find the positive square root of the base 7 number  $(551662)_7$ , giving your answer as a base 7 number.

## Markscheme

converting to base 10

$$(551662)_7 = 2 + 6 \times 7 + 6 \times 7^2 + 1 \times 7^3 + 5 \times 7^4 + 5 \times 7^5 \quad (M1)$$

$$= 96721 \quad (A1)$$

$$\sqrt{96721} = 311 \quad A1$$

converting back to base 7

$$7 \overline{)311} \quad (M1)$$

$$\underline{)44} (3)$$

$$\underline{)6} (2) \quad (A1)$$

$$\text{it follows that } \sqrt{(551662)_7} = (623)_7 \quad A1$$

**Note:** Accept 623.

[6 marks]

## Examiners report

[N/A]

---

(a) Show that the solution to the linear congruence  $ax \equiv b \pmod{p}$ , where  $a, x, b, p \in \mathbb{Z}^+$ ,  $p$  is prime and  $a, p$  are relatively prime, is given

by  $x \equiv a^{p-2}b \pmod{p}$ .

(b) Consider the congruences

$$7x \equiv 13 \pmod{19}$$

$$2x \equiv 1 \pmod{7}.$$

(i) Use the result in (a) to solve the first congruence, giving your answer in the form  $x \equiv k \pmod{19}$  where  $1 \leq k \leq 18$ .

(ii) Find the set of integers which satisfy both congruences simultaneously.

## Markscheme

(a) using Fermat's little theorem,  $(M1)$

$$a^{p-1} \equiv 1 \pmod{p} \quad A1$$

multiplying both sides of the congruence by  $a^{p-2}$ ,  $(M1)$

$$a^{p-1}x \equiv a^{p-2}b \pmod{p} \quad \text{AI}$$

$$x \equiv a^{p-2}b \pmod{p} \quad \text{AG}$$

[4 marks]

(b) (i) the solution is

$$x \equiv 7^{17} \times 13 \pmod{19} \quad \text{AI}$$

consider

$$7^3 = 343 \equiv 1 \pmod{19} \quad (\text{AI})$$

**Note:** Other powers are possible.

therefore

$$x \equiv (7^3)^5 \times 7^2 \times 13 \pmod{19} \quad (\text{AI})$$

$$\equiv 7^2 \times 13 \pmod{19} \quad (\text{AI})$$

$$\equiv 10 \pmod{19} \quad \text{AI}$$

(ii) using any method, including trial and error, the solution to the second congruence is given by  $x \equiv 32 \pmod{7}$  (or equivalent) (AI)

a simultaneous solution is  $x = 67$  (or equivalent, eg  $-66$ ) AI

the full solution is  $x = 67 + 133N$  (where  $N \in \mathbb{Z}$ ) (or equivalent) AI

**Note:** Do not *FT* an incorrect answer from (i).

[8 marks]

## Examiners report

[N/A]

Given that  $n^2 + 2n + 3 \equiv N \pmod{8}$ , where  $n \in \mathbb{Z}^+$  and  $0 \leq N \leq 7$ , prove that  $N$  can take one of only three possible values.

## Markscheme

consider the following

$n$	$(n^2 + 2n + 3) \pmod{8}$
1	6
2	3
3	2
4	3
5	6
6	3
7	2
8	3

*MIAIAI*

we see that the only possible values so far are 2, 3 and 6 **RI**

also, the table suggests that these values repeat themselves but we have to prove this

let  $f(n) = n^2 + 2n + 3$ , consider

$$f(n+4) - f(n) = (n+4)^2 + 2(n+4) + 3 - n^2 - 2n - 3 \quad \text{MI}$$

$$= 8n + 24 \quad \text{AI}$$

since  $8n + 24$  is divisible by 8, **MI**

$$f(n + 4) \equiv (n) \pmod{8} \quad \text{AI}$$

this confirms that the values do repeat every 4 values of  $n$  so that 2, 3 and 6 are the only values taken for all values of  $n$  **RI**

[9 marks]

## Examiners report

Solutions to this question were extremely disappointing in general. Many candidates spotted that there was a periodicity in the values of the polynomial as far as  $n = 4$  and even  $n = 8$  but then the majority simply assumed that this would continue without any need for proof. This is equivalent to noting that  $2^{2n+1} - 1$  is prime for  $n = 1, 2$  and  $3$  and simply assuming that this will be true for larger values of  $n$ .

---

A sequence  $\{u_n\}$  satisfies the recurrence relation  $u_{n+2} = 2u_{n+1} - 5u_n$ ,  $n \in \mathbb{N}$ . Obtain an expression for  $u_n$  in terms of  $n$  given that  $u_0 = 0$  and  $u_1 = 1$ .

## Markscheme

the auxiliary equation is  $m^2 - 2m + 5 = 0$  **MI**

the roots are  $1 \pm 2i$  **AI**

the general solution is  $u_n = A(1 + 2i)^n + B(1 - 2i)^n$  **AI**

substituting  $u_0 = 0$  and  $u_1 = 1$  **MI**

$$A + B = 0$$

$$A(1 + 2i) + B(1 - 2i) = 1 \quad \text{AI}$$

**Note:** Only award above **AI** if both equations are correct.

solving

$$A(1 + 2i - 1 + 2i) = 1 \quad \text{(MI)}$$

$$A = \frac{1}{4i} \text{ or } -\frac{i}{4} \quad \text{AI}$$

$$B = -\frac{1}{4i} \text{ or } \frac{i}{4} \quad \text{AI}$$

therefore

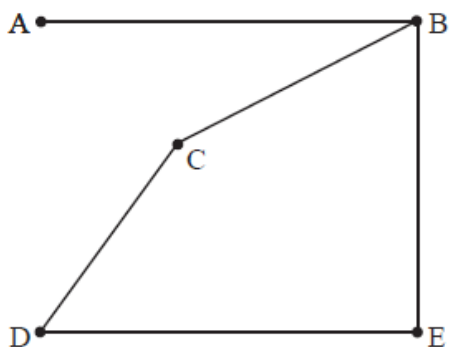
$$u_n = \frac{1}{4i}(1 + 2i)^n - \frac{1}{4i}(1 - 2i)^n \text{ or } \frac{i}{4}(1 - 2i)^n - \frac{i}{4}(1 + 2i)^n \quad \text{AI}$$

[9 marks]

## Examiners report

[N/A]

The figure below shows the graph  $G$ .



a. (i) Write down the adjacency matrix for  $G$ .

[5]

(ii) Find the number of walks of length 4 beginning and ending at B.

b. (i) Draw  $G'$ , the complement of  $G$ .

[6]

(ii) Write down the degrees of all the vertices of  $G$  and all the vertices of  $G'$ .

(iii) Hence, or otherwise, determine whether or not  $G$  and  $G'$  are isomorphic.

## Markscheme

a. (i) the adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad A2$$

**Note:** Award  $A2$  for correct matrix,  $A1$  for one or two errors and  $A0$  for more than two errors.

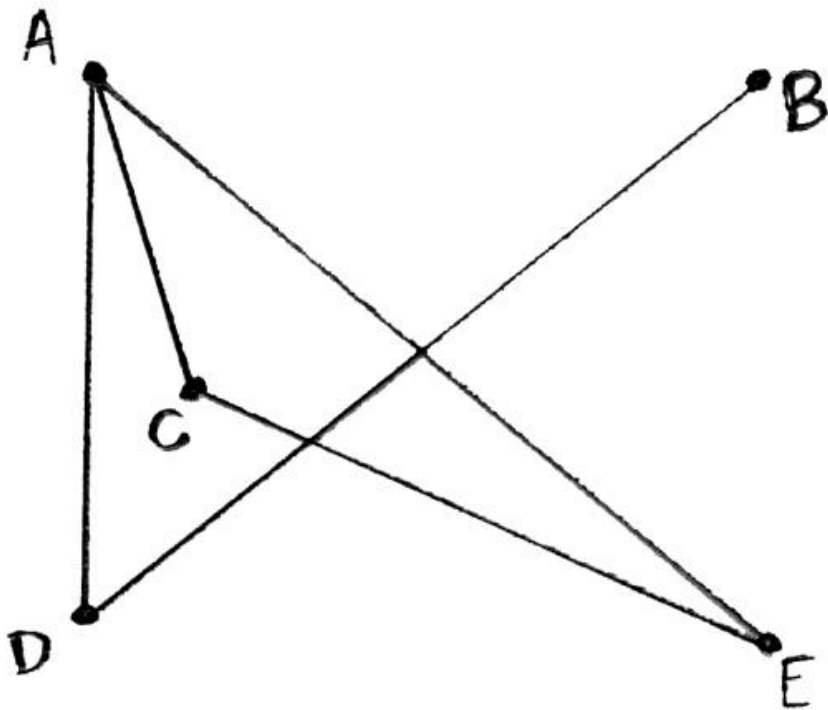
(ii) the required number of walks is the (B, B) element in the fourth power of the adjacency matrix **(M1)**

using a GDC gives the answer as 13 **A2**

**[5 marks]**

b. (i)





A2

(ii)

	$G$	$G'$
<b>A</b>	1	3
<b>B</b>	3	1
<b>C</b>	2	2
<b>D</b>	2	2
<b>E</b>	2	2

A1A1

(iii) In  $G$ , the vertex of degree 1 is adjacent to the vertex of degree 3, whereas in  $G'$  the vertex of degree 1 is adjacent to a vertex of degree 2. They are not therefore isomorphic. **A1R1**

**Note:** Accept alternative correct solutions.

[6 marks]

## Examiners report

- Many candidates wrote down the adjacency matrix correctly and then proceeded to raise it to the fourth power, although at least one candidate actually enumerated, correctly, the 13 walks.
- Part (b) was reasonably well done with many candidates realising that, although the degrees of the vertices were the same, their relative positions meant that the two graphs were not isomorphic.

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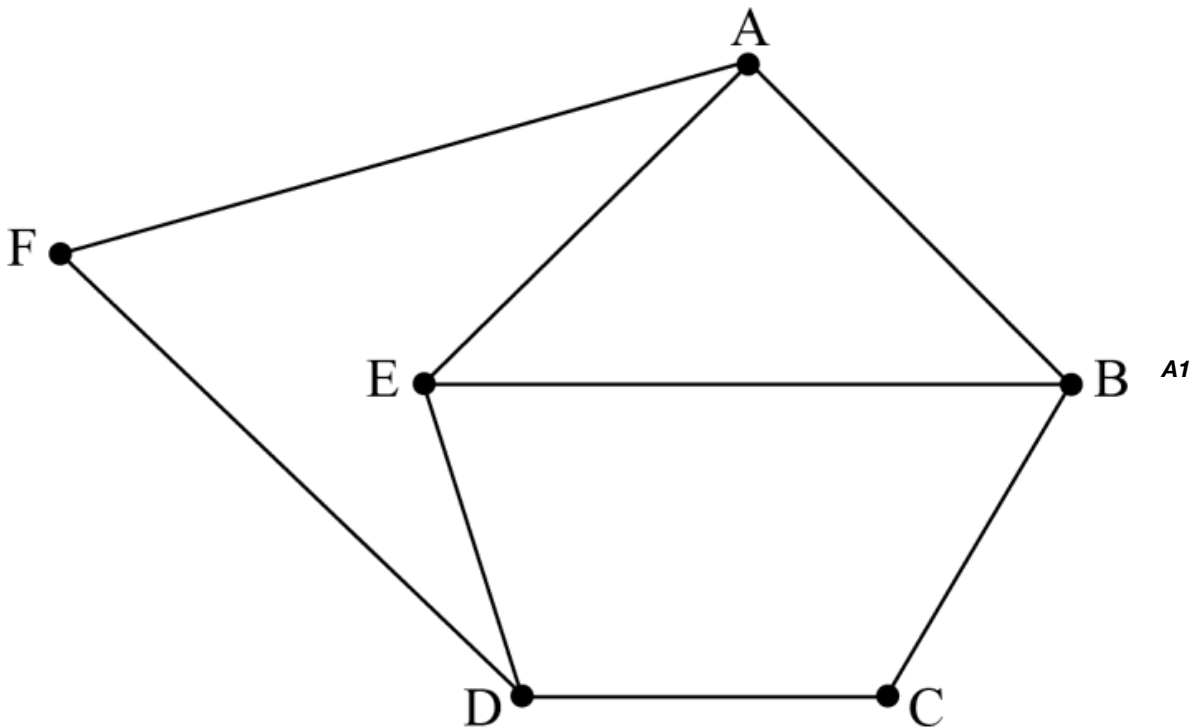
A simple graph  $G$  is represented by the following adjacency table.

	A	B	C	D	E	F
A	-	1	-	-	1	1
B	1	-	1	-	1	-
C	-	1	-	1	-	-
D	-	-	1	-	1	1
E	1	1	-	1	-	-
F	1	-	-	1	-	-

- a. Draw the simple graph  $G$ . [1]
- b. Explain why  $G$  does not contain an Eulerian circuit. [1]
- c. Show that  $G$  has a Hamiltonian cycle. [2]
- d. State whether or not  $G$  is planar, giving a reason for your answer. [2]
- e. State whether or not the simple graph  $G$  is bipartite, giving a reason for your answer. [2]
- f. Draw the complement  $G'$  of  $G$ . [2]

## Markscheme

a.



b. because it has vertices which are not of even degree. **A1**

c. for example  $D \rightarrow C \rightarrow B \rightarrow E \rightarrow A \rightarrow F \rightarrow D$  **A2**

d.  $G$  is planar. **A1**

either note from original graph in part (a) that there are no edges crossing or a redrawing with statement that there are no edges crossing. **R1**

**Note:** Do not accept an argument based on  $e \leq 3v - 6$

e. the graph is not bipartite. **A1**

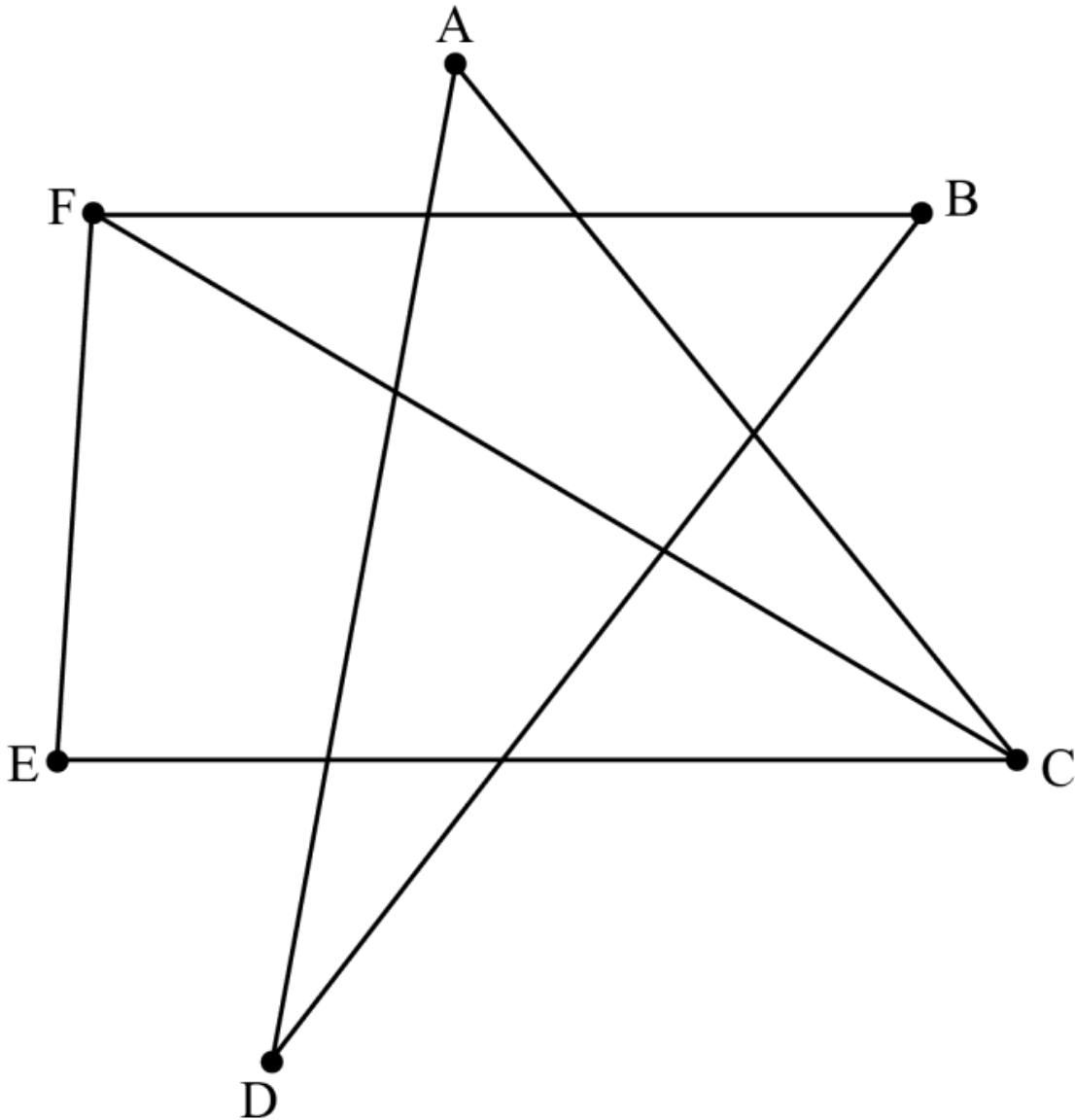
**EITHER**

the graph contains a triangle **R1**

**OR**

to be bipartite according to line 1 of the table A, C and D need to be one set as A connects to B, E and F. Since C and D are connected, this is contradicted. **R1**

f.



**A2**

**Note:** Award **A1** if extra line seen or missed out.

## Examiners report

a. This question was very well answered in general.

- b. This question was very well answered in general.
  - c. This question was very well answered in general.
  - d. This question was very well answered in general. In (d), some candidates stated that  $G$  is planar because  $e \leq 3v - 6$ . It is however important to realise that this condition is necessary for a graph to be planar but not sufficient. Some candidates stated that  $G$  is planar 'because I have drawn it as a planar graph' or even 'see graph in (a)'. Candidates were expected to state that  $G$  is planar because it can be drawn with no edges crossing.
  - e. This question was very well answered in general.
  - f. This question was very well answered in general.
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